Advances in Analysis, Vol. 3, No. 3, July 2018
https://dx.doi.org/10.22606/aan.2018.33001

Some Improvements of Ambarzumyan's Theorem

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Abstract We extend the classical Ambarzumyan's theorem to the quasi-periodic boundary value problems by using only a part knowledge of one spectrum. We also weaken slightly the Yurko's conditions on the first eigenvalue.

Keywords: Ambarzumyan theorem, inverse spectral theory, Hill operator, quasi-periodic boundary conditions

1 Introduction and Preliminaries

In 1929, Ambarzumyan [1] proved that if $\{(n\pi)^2 : n = 0, 1, 2 \ldots\}$ is the spectrum of the boundary value problem

$$-y''(x) + q(x)y(x) = \lambda y(x), \qquad y'(0) = y'(1) = 0 \tag{1}$$

with real potential $q \in L^1[0,1]$, then $q = 0$ a.e. Clearly, if $q = 0$ a.e., then the eigenvalues $\lambda_n = (n\pi)^2$, $n \geq 0$.

Freiling and Yurko [6] proved that it is enough to specify only the first eigenvalue rather than the whole spectrum. More precisely, the first eigenvalue denoted by λ_0 is a mean value of the potential, that is, they proved the following Ambarzumyan-type theorem:

Theorem 1.1 If $\lambda_0 = \int_0^1 q(x)\,dx$, then $q = \lambda_0$ a.e.

In [13], Yurko also provided generalizations of Theorem 1.1 on wide classes of self-adjoint differential operators. Some of the inverse results in [13] are as follows:

Theorem 1.2 (a) Let

$$\lambda_0 = \int_0^1 q(x)\,dx$$

be the first eigenvalue of the periodic boundary value problem

$$-y''(x) + q(x)y(x) = \lambda y(x), \qquad y(1) = y(0), \quad y'(1) = y'(0), \tag{2}$$

then $q = \lambda_0$ a.e.

(b) Let

$$\lambda_0 = \pi^2 + \frac{2}{\alpha^2+\beta^2}\int_0^1 q(x)(\alpha sin\pi x + \beta cos\pi x)^2\,dx, \tag{3}$$

for some fixed α and β, be the first eigenvalue of the anti-periodic boundary value problem

$$-y''(x) + q(x)y(x) = \lambda y(x), \qquad y(1) = -y(0), \quad y'(1) = -y'(0). \tag{4}$$

Then

$$q = \frac{2}{\alpha^2+\beta^2}\int_0^1 q(x)(\alpha sin\pi x + \beta cos\pi x)^2\,dx \quad a.e.$$

Consider the boundary value problems $L_t(q)$ generated in the space $L^2[0,1]$ by the following differential equation and quasi-periodic boundary conditions

$$-y''(x) + q(x)y(x) = \lambda y(x), \qquad y(1) = e^{it}y(0), \quad y'(1) = e^{it}y'(0), \tag{5}$$

where $q \in L^1[0,1]$ is a real-valued function and $t \in [0,\pi) \cup [\pi, 2\pi)$. The operator $L_t(q)$ is self-adjoint and the cases $t = 0$ and $t = \pi$ correspond to the periodic and anti-periodic problems, respectively. Let $\{\lambda_n(t)\}_{n\in\mathcal{Z}}$ be the eigenvalues of the operator $L_t(q)$. In the case when $q = 0$, $(2\pi n + t)^2$ for $n \in \mathcal{Z}$ is the eigenvalue of the operator $L_t(0)$ for any fixed $t \in [0, 2\pi)$ corresponding to the eigenfunction $e^{i(2\pi n+t)x}$.

The result of Ambarzumyan [1] is an exceptional situation to the general rule. In general, Borg [2] proved that one spectrum does not determine the potential. Also, Borg showed that two spectra determine it uniquely. In [5], by imposing an additional condition on the potential they extended the classical Ambarzumyan's theorem for the Sturm-Liouville equation to the general separated boundary conditions. Many generalizations of the Ambarzumyan's theorem can be found in [3,4,7,8,12]. As far as we know, for the first time without using any additional conditions on the potential, we extend the classical Ambarzumyan theorem to the quasi-periodic boundary value problems $L_t(q)$ by using only a part knowledge of one spectrum (see Theorem 1.4). More precisely, the aim of this paper is to weaken slightly the conditions in Theorem 1.1 and Theorem 1.2 for the first eigenvalue (see Theorem 1.3) and to prove a new generalization of Ambarzumyan theorem. We also extend Theorem 1.1 of the previous paper [9].

Our new uniqueness-type results read as follows:

Theorem 1.3 Let $\lambda_0(t)$ be the first eigenvalue of $L_t(q)$. Then:
(a) If $\lambda_0(t) \geq t^2 + \int_0^1 q(x)\,dx$ for $t \in [0,\pi)$, then $q = \int_0^1 q(x)\,dx$ a.e.
(b) If $\lambda_0(t) \geq (2\pi - t)^2 + \int_0^1 q(x)\,dx$ for $t \in [\pi, 2\pi)$, then $q = \int_0^1 q(x)\,dx$ a.e.

Theorem 1.4 Let $\lambda_0(t)$ be the first eigenvalue of $L_t(q)$, and suppose that n_0 is a sufficiently large positive integer. Then the following two assertions hold:
(a) If $\lambda_0(t) \geq t^2$ and $\lambda_n(t) = (2n\pi + t)^2$ for $t \in [0,\pi)$ and $n > n_0$, then $q = 0$ a.e.
(b) If $\lambda_0(t) \geq (2\pi - t)^2$ and $\lambda_n(t) = (2n\pi + t)^2$ for $t \in [\pi, 2\pi)$ and $n > n_0$, then $q = 0$ a.e.

In Theorem 1.3 (a), the case $t = 0$ corresponds to the periodic problem (2) and the assertion of Theorem 1.2 (a) holds by using $\lambda_0(0) \geq \int_0^1 q(x)\,dx$ instead of $\lambda_0(0) = \int_0^1 q(x)\,dx$. In Theorem 1.3 (b), the case $t = \pi$ corresponds to the anti-periodic problem (4) and the assertion of Theorem 1.3 (b) holds by using $\lambda_0(\pi) \geq \pi^2 + \int_0^1 q(x)\,dx$ instead of (3). Namely, unlike Theorem 1.2 (b), the first eigenvalue in Theorem 1.3 (b) depends only on the mean value of the potential as in Theorem 1.1 and Theorem 1.2 (a). Whether the first eigenvalue-types of Ambarzumyan theorems always depend on a mean value of the potential can be investigated in another paper.

And, for the boundary value problem (1), the form with reduced spectrum of Ambarzumyan's theorem read as follows:

Theorem 1.5 (a) If $\lambda_0 \geq \int_0^1 q(x)\,dx$, then $q = \int_0^1 q(x)\,dx$ a.e.
(b) If $\lambda_0 \geq 0$ and $\lambda_n = (n\pi)^2$ for $n > n_0$, then $q = 0$ a.e., where n_0 is a sufficiently large positive integer.

Note that, for example, in Theorem 1 if the first eigenvalue $\lambda_0 = 0$ and $\int_0^1 q(x)\,dx = 0$, then $q = 0$ a.e. Hence, to prove Theorem 1.4 and Theorem 1.5 (b), without imposing an additional condition on the potential such as $\int_0^1 q(x)\,dx = 0$, we have information about the first eigenvalue with a less restrictive one and a subset of the sufficiently large eigenvalues of the spectrum.

2 Proofs

Proof of Theorem 1.3. (a) We show that $y = e^{itx}$ is the first eigenfunction corresponding to the first eigenvalue $\lambda_0(t)$ of the operator $L_t(q)$ for $t \in [0,\pi)$. Since the test function $y = e^{itx}$ satisfies the quasi-periodic boundary conditions in (5), by the variational principle, we get

$$t^2 + \int_0^1 q(x)\,dx \leq \lambda_0(t) \leq \frac{\int_0^1 -\bar{y}y''dx + \int_0^1 q(x)|y|^2dx}{\int_0^1 |y|^2dx} = t^2 + \int_0^1 q(x)\,dx. \tag{6}$$

This implies that $\lambda_0(t) = (t^2 + \int_0^1 q(x)\,dx)$ is the first eigenvalue corresponding to the first eigenfunction $y = e^{itx}$. Substituting this into the equation

$$-y'' + q(x)y = \lambda y,$$

we get $q = \int_0^1 q(x)\,dx$ a.e.

(b) Similarly, for $t \in [\pi, 2\pi)$, the test function $y = e^{i(-2\pi+t)x}$ is the first eigenfunction corresponding to the first eigenvalue $\lambda_0(t) = (2\pi - t)^2 + \int_0^1 q(x)\,dx$. Thus, $q = \int_0^1 q(x)\,dx$ a.e. □

Proof of Theorem 1.4. Note that by [10] (see also [11]), without using the assumption $\int_0^1 q(x)\,dx = 0$, the eigenvalues $\lambda_n(t)$ of the operator $L_t(q)$ for $t \neq 0, \pi$ are asymptotically located such that

$$\lambda_n(t) = (2\pi n + t)^2 + \int_0^1 q(x)\,dx + O\left(n^{-1} ln|n|\right), \tag{7}$$

for $|n| \geq n_0$, where n_0 is a sufficiently large positive integer. And, by Theorem 1.1 of [3], there is a similar asymptotic formulas for the sufficiently large eigenvalues $\lambda_n(t)$ of the operator $L_t(q)$ for $t = 0, \pi$. Thus, for all $t \in [0, \pi) \cup [\pi, 2\pi)$, if $\lambda_n(t) = (2n\pi + t)^2$ for $n > n_0$, then $\int_0^1 q(x)\,dx = 0$. From Theorem 1.3, $q = 0$ a.e. Thus Theorem 1.4 (a) and (b) are proved. □

Remark. Note that, in Theorem 1.4 (a) and (b), if we use the large eigenvalues $\lambda_{-n}(t) = (2n\pi - t)^2$ of the spectrum $S(L_t(q))$ instead of $\lambda_n(t) = (2n\pi + t)^2$, then the assertions of Theorem 1.4 remain valid. Thus, to prove the theorem for all $|n| > n_0$, it is enough to set either of the eigenvalues $\lambda_n(t)$, $\lambda_{-n}(t)$.

Proof of Theorem 1.5. (a) Arguing as in the proof of Theorem 1.3 (a), for $\lambda_0 \geq \int_0^1 q(x)\,dx$, the test function $y = 1$ is the first eigenfunction corresponding to the first eigenvalue $\lambda_0 = \int_0^1 q(x)\,dx$. Thus, $q = \int_0^1 q(x)\,dx$ a.e.

(b) It follows from the asymptotic formula (1.6.6) in [8] that if $\lambda_n = (n\pi)^2$ for $n > n_0$, then certainly $\int_0^1 q(x)\,dx = 0$. From (a), $q = 0$ a.e. □

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